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Research Article

A Convexity Property for an Integral Operator on the Class $S_p(\beta)$

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We consider an integral operator, $F_n(z)$, for analytic functions, $f_i(z)$, in the open unit disk, U . The object of this paper is to prove the convexity properties for the integral operator $F_n(z)$, on the class $S_p(\beta)$.

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1. Introduction

Let $U = \{z \in \mathbb{C}, |z| < 1\}$ be the unit disc of the complex plane and denote by $H(U)$ the class of the holomorphic functions in U . Let $A = \{f \in H(U), f(z) = z + a_2 z^2 + a_3 z^3 + \dots, z \in U\}$ be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$.

Denote with K the class of convex functions in U , defined by

$$K = \left\{ f \in A : \operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0, z \in U \right\}. \quad (1.1)$$

A function $f \in S$ is the convex function of order α , $0 \leq \alpha < 1$, and denote this class by $K(\alpha)$ if f verifies the inequality

$$\operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > \alpha, z \in U. \quad (1.2)$$

Consider the class $S_p(\beta)$, which was introduced by Ronning [1] and which is defined by

$$f \in S_p(\beta) \iff \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \beta \right\}, \quad (1.3)$$

where β is a real number with the property $-1 \leq \beta < 1$.

For $f_i(z) \in A$ and $\alpha_i > 0$, $i \in \{1, \dots, n\}$, we define the integral operator $F_n(z)$ given by

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\alpha_1} \dots \left(\frac{f_n(t)}{t} \right)^{\alpha_n} dt. \quad (1.4)$$

This integral operator was first defined by B. Breaz and N. Breaz [2]. It is easy to see that $F_n(z) \in A$.

2. Main results

Theorem 2.1. Let $\alpha_i > 0$, for $i \in \{1, \dots, n\}$, let β_i be real numbers with the property $-1 \leq \beta_i < 1$, and let $f_i \in S_p(\beta_i)$ for $i \in \{1, \dots, n\}$.

If

$$0 < \sum_{i=1}^n \alpha_i (1 - \beta_i) \leq 1, \quad (2.1)$$

then the function F_n given by (1.4) is convex of order $1 + \sum_{i=1}^n \alpha_i (\beta_i - 1)$.

Proof. We calculate for F_n the derivatives of first and second orders.

From (1.4) we obtain

$$\begin{aligned} F'_n(z) &= \left(\frac{f_1(z)}{z} \right)^{\alpha_1} \dots \left(\frac{f_n(z)}{z} \right)^{\alpha_n}, \\ F''_n(z) &= \sum_{i=1}^n \alpha_i \left(\frac{f_i(z)}{z} \right)^{\alpha_i} \left(\frac{zf'_i(z) - f_i(z)}{zf_i(z)} \right) \prod_{\substack{j=1 \\ j \neq i}}^n \left(\frac{f_j(z)}{z} \right)^{\alpha_j}. \end{aligned} \quad (2.2)$$

After some calculus, we obtain that

$$\frac{F''_n(z)}{F'_n(z)} = \alpha_1 \left(\frac{zf'_1(z) - f_1(z)}{zf_1(z)} \right) + \dots + \alpha_n \left(\frac{zf'_n(z) - f_n(z)}{zf_n(z)} \right). \quad (2.3)$$

This relation is equivalent to

$$\frac{F''_n(z)}{F'_n(z)} = \alpha_1 \left(\frac{f'_1(z)}{f_1(z)} - \frac{1}{z} \right) + \dots + \alpha_n \left(\frac{f'_n(z)}{f_n(z)} - \frac{1}{z} \right). \quad (2.4)$$

If we multiply the relation (2.4) with z , then we obtain

$$\frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^n \alpha_i \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right) = \sum_{i=1}^n \alpha_i \frac{zf'_i(z)}{f_i(z)} - \sum_{i=1}^n \alpha_i. \quad (2.5)$$

The relation (2.5) is equivalent to

$$\frac{zF''_n(z)}{F'_n(z)} + 1 = \sum_{i=1}^n \alpha_i \frac{zf'_i(z)}{f_i(z)} - \sum_{i=1}^n \alpha_i + 1. \quad (2.6)$$

This relation is equivalent to

$$\frac{zF_n''(z)}{F_n'(z)} + 1 = \sum_{i=1}^n \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - \beta_i \right) + \sum_{i=1}^n \alpha_i \beta_i - \sum_{i=1}^n \alpha_i + 1. \quad (2.7)$$

We calculate the real part from both terms of the above equality and obtain

$$\operatorname{Re} \left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) = \sum_{i=1}^n \alpha_i \operatorname{Re} \left(\frac{zf_i'(z)}{f_i(z)} - \beta_i \right) + \sum_{i=1}^n \alpha_i \beta_i - \sum_{i=1}^n \alpha_i + 1. \quad (2.8)$$

Because $f_i \in S_p(\beta_i)$ for $i = \{1, \dots, n\}$, we apply in the above relation inequality (1.3) and obtain

$$\operatorname{Re} \left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) > \sum_{i=1}^n \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \sum_{i=1}^n \alpha_i (\beta_i - 1) + 1. \quad (2.9)$$

Since $\alpha_i |zf_i'(z)/f_i(z) - 1| > 0$ for all $i \in \{1, \dots, n\}$, we obtain that

$$\operatorname{Re} \left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) > \sum_{i=1}^n \alpha_i (\beta_i - 1) + 1. \quad (2.10)$$

So, F_n is convex of order $\sum_{i=1}^n \alpha_i (\beta_i - 1) + 1$. \square

Corollary 2.2. Let α_i , $i \in \{1, \dots, n\}$ be real positive numbers and $f_i \in S_p(\beta)$ for $i \in \{1, \dots, n\}$.
If

$$0 < \sum_{i=1}^n \alpha_i \leq \frac{1}{1 - \beta}, \quad (2.11)$$

then the function F_n is convex of order $(\beta - 1) \sum_{i=1}^n \alpha_i + 1$.

Proof. In Theorem 2.1, we consider $\beta_1 = \beta_2 = \dots = \beta_n = \beta$. \square

Remark 2.3. If $\beta = 0$ and $\sum_{i=1}^n \alpha_i = 1$, then

$$\operatorname{Re} \left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) > 0, \quad (2.12)$$

so F_n is a convex function.

Corollary 2.4. Let γ be a real number, $\gamma > 0$. Suppose that the functions $f \in S_p(\beta)$ and $0 < \gamma \leq 1/(1 - \beta)$. In these conditions, the function $F_1(z) = \int_0^z (f(t)/t)^\gamma dt$ is convex of order $(\beta - 1)\gamma + 1$.

Proof. In Corollary 2.2, we consider $n = 1$. \square

Corollary 2.5. Let $f \in S_p(\beta)$ and consider the integral operator of Alexander, $F(z) = \int_0^z (f(t)/t) dt$. In this condition, F is convex by the order β .

Proof. We have

$$\frac{zF''(z)}{F'(z)} = \frac{zf'(z)}{f(z)} - 1. \quad (2.13)$$

From (2.13), we have

$$\operatorname{Re}\left(\frac{zF''(z)}{F'(z)} + 1\right) = \operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \beta\right) + \beta > \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta > \beta. \quad (2.14)$$

So, the relation (2.14) implies that the Alexander operator is convex. \square

References

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